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A Survey of Four-Dimensional $\mathbb{C}$-Associative Algebras

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A four-dimensional real algebra $\mathfrak{A}$ containing a subalgebra $\mathfrak{C}$ isomorphic to the complex numbers is left $\mathbb{C}$-associative (with respect to $\mathfrak{C}$) if $\mathfrak{A}$ is a bimodule with respect to $\mathfrak{C}$ and $L(MR) = (LM)R$ whenever L is in $\mathfrak{C}$. (Middle and right $\mathbb{C}$-associativity are defined similarly.) This article contains a survey of results about algebras having one or more $\mathbb{C}$-associative properties.

1. INTRODUCTION

In 1843, William Rowan Hamilton invented the four-dimensional associative real algebra he called the quaternions. He had originally hoped to find a three-dimensional analog of the complex number system, but as was later proved, none exists. As is well-known, Hamilton's creation of the quaternions required a move away from the security of the commutative law of multiplication. As Frobenius (and two others independently) proved, the inventive process ends if there are no further weakenings of general laws about multiplication: the only four-dimensional associative real division algebra is the quaternions [13], [10], [15].

However, if one is willing to weaken associativity, new opportunities open up. Assumptions of alternativity, *i.e.* $(xx)y = x(xy)$ and $x(yy) = (xy)y$ or power associativity, *i.e.* $x(x^2) = x^2x$ and $(x^2x)x = (x^2)^2$ [1] were among the first to lead to interesting new non-associative algebras [14]. In this paper, we survey a relatively recently studied class of four-dimensional real algebras: those which contain a copy of the complex numbers and satisfy what we call $\mathbb{C}$-associative properties. These properties produce relatively simple multiplication tables for the basis elements of the algebras. In turn, the tables produce a framework for the classification of the algebras in various ways: isomorphism classes, derivation algebras, and number of generators.

DEFINITIONS [6]. Let $\mathfrak{A}$ be an algebra over $\mathbb{R}$ containing an isomorphic copy $\mathfrak{C}$ of the complex numbers with respect to which $\mathfrak{A}$ is a $\mathfrak{C}$-*bimodule*; that is, such that L(MR) = (LM)R whenever any two of L, M or R are in $\mathfrak{C}$. The algebra $\mathfrak{A}$ is called

(i) *left $\mathbb{C}$-associative* if L(MR) = (LM)R, for all L in $\mathfrak{C}$ and M, R in $\mathfrak{A}$;

(ii) *middle $\mathbb{C}$-associative* if L(MR) = (LM)R, for all M in $\mathfrak{C}$ and L, R in $\mathfrak{A}$;

(iii) *right $\mathbb{C}$-associative* if (LM)R = L(MR), for all R in $\mathfrak{C}$ and L, M in $\mathfrak{A}$.

We consider three types of $\mathbb{C}$-associative algebras:

(a) *strictly left* (or *middle* or *right*)*$\mathbb{C}$-associative algebras* which satisfy only one of the corresponding conditions (i) - (iii);

(b) *two-$\mathbb{C}$-associative algebras* which satisfy exactly two of the conditions;

(c) *fully-$\mathbb{C}$-associative algebras* which satisfy all three conditions.

If a finite-dimensional real algebra $\mathfrak{A}$ contains a copy $\mathfrak{C}$ of the complex numbers and $\mathfrak{A}$ is a $\mathfrak{C}$-bimodule, then because $\mathfrak{A}$ is a vector space over $\mathbb{C}$, $\mathfrak{A}$ must be even-dimensional over $\mathbb{R}$. **For the rest of this paper, $\mathfrak{A}$ is such an algebra of real dimension four with subalgebra $\mathfrak{C}$.** We denote the identity element of $\mathfrak{C}$ (and $\mathfrak{A}$) by 1, identify $\mathbb{R}$ with $\mathbb{R}\bullet 1$ in $\mathfrak{A}$, and let i in $\mathfrak{C}$ satisfy $i^2 = -1$.

The first result, which depends only on the assumption that $\mathfrak{A}$ is a $\mathfrak{C}$-bimodule, gives a multiplication table for $\mathfrak{A}$ in which the first two rows and columns are very simple.

PROPOSITION 1.1. If J lies in $\mathbb{A} - \mathbb{C}$, then $\{1, i, J, iJ\}$ is a basis for $\mathbb{A}$ over $\mathbb{R}$. Furthermore, either $\mathbb{A}$ is commutative or there exists a J with $iJ + Ji = 0$.

Proof. See [3].

THE COMMUTATIVE CASE

One can obtain a complete classification of commutative algebras $\mathbb{A}$ into one of three types.

PROPOSITION 1.2. If $\mathbb{A}$ is both commutative and left (or right) $\mathbb{C}$-associative, then $\mathbb{A}$ is fully-$\mathbb{C}$-associative and has a basis $\{1, i, J, iJ\}$ with respect to which it has exactly one of the following two tables:

(S_1)

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	-J
J	J	iJ	0	0
iJ	iJ	-J	0	0

(S_2)

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	-J
J	J	iJ	-1	-1
iJ	iJ	-J	-1	1

Proof. See Lemma 2 in [6] and Theorem 3 in [3].

The algebra with table (S_1) turns out to be Peirce's associative algebra $[1^2]$. (See [15].) The algebra with table (S_2) is not associative. However, it is $\mathbb{C}$-associative with respect to exactly two distinct subalgebras with respective bases $\{1, i\}$ and $\{1, J\}$.

THEOREM 1.3. If $\mathfrak{A}$ is commutative and strictly middle $\mathbb{C}$-associative, then $\mathfrak{A}$ has a table of the form

(T_c)

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	-J
J	J	iJ	a + bi	f + gi + hJ
iJ	iJ	-J	f + gi + hJ	-a - bi

where h is either 0 or 1. Such an algebra is isomorphic to another with corresponding parameters a', b', f', g', h' if and only if either

(i) $h = h' = 0$ and there exist $r > 0$ and θ such that

$$\begin{bmatrix} a & b \\ f & g \end{bmatrix} = rR_\theta \begin{bmatrix} a' & \pm b' \\ \pm f' & g' \end{bmatrix}$$

or (ii) $h = h' = 1$ and

$$\begin{bmatrix} a & b \\ f & g \end{bmatrix} = \begin{bmatrix} a' & \pm b' \\ \pm f' & g' \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} a & b \\ f & g \end{bmatrix} = \begin{bmatrix} -\frac{3}{32} & \frac{3\varepsilon\sqrt{3}}{32} \\ \frac{\varepsilon\sqrt{3}}{32} & \frac{9}{32} \end{bmatrix} + R_{\frac{2\pi\varepsilon}{3}} \begin{bmatrix} a' & \pm b' \\ \pm f' & g' \end{bmatrix}, \text{ where } \varepsilon^2 = 1.$$

Here R_α denotes the rotation matrix $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

Proof. See [5].

By Lemma 1.2 in [5], a strictly middle $\mathbb{C}$-associative commutative algebra has exactly one two-dimensional subalgebra with respect to which it is middle $\mathbb{C}$-associative.

THE NONCOMMUTATIVE CASE

In the noncommutative case, the classification of $\mathbb{C}$-associative algebras with respect to isomorphism types is more intricate. It involves by identifying canonical multiplication tables for each type.

PROPOSITION 1.4. If $\mathfrak{A}$ is a noncommutative $\mathbb{C}$-bimodule, then $\mathfrak{A}$ has a basis $\{1, i, J, iJ\}$ such that $Jc = \overline{c}\, J$ for every $c \in \mathbb{C}$, where $\overline{c}$ denotes the usual complex conjugate of c. Thus, $\mathfrak{A}$ has a table of the form

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	-J
J	J	-iJ	*	*
iJ	iJ	J	*	*

Proof. See Corollary 5 in [3].

If $(\mathfrak{A}, \bullet)$ is an algebra, the opposite algebra $(\mathfrak{A}^{\mathrm{opp}}, \circ)$ is the set of elements of $\mathfrak{A}$ with the operation $\circ$ defined by $A \circ B = B \bullet A$. Clearly, $(\mathfrak{A}, \bullet)$ is right $\mathbb{C}$-associative if and only if $(\mathfrak{A}^{\mathrm{opp}}, \circ)$ is left $\mathbb{C}$-associative. Note that if $\{1, i, J, i \bullet J\}$ is a basis for $\mathfrak{A}$ then $\{1, i, J, i \circ J\}$ is a basis for $\mathfrak{A}^{\mathrm{opp}}$. Because of these facts it is generally unnecessary to study right $\mathbb{C}$-associative algebras directly. For left and middle $\mathbb{C}$-associative algebras, the multiplication tables are given explicitly as follows.

THEOREM 1.5. $\mathfrak{A}$ is noncommutative and left $\mathbb{C}$-associative if and only if it has a table of the form

(T_ℓ)

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	-J
J	J	-iJ	a + bi + cJ + diJ	f + gi + hJ + kiJ
iJ	iJ	J	-b + ai - dJ + ciJ	-g + fi - kJ + hiJ

$\mathfrak{A}$ is noncommutative and middle $\mathbb{C}$-associative if and only if it has a table of the form

(T_m)

	1	i	J	iJ
1	1	i	J	iJ
i	i	-1	iJ	-J
J	J	-iJ	a + bi + cJ + diJ	f + gi + hJ + kiJ
iJ	iJ	J	-f - gi - hJ - kiJ	a + bi + cJ + diJ

Proof. See [4] and [5].

When an algebra satisfies more than one level of $\mathbb{C}$-associativity, its table takes a more specialized form:

THEOREM 1.6. An algebra $\mathfrak{A}$ with Table (T_ℓ) is

(i) left and right $\mathbb{C}$-associative with respect to span{1, i} if and only if

$$f = b,\ g = -a,\ h = -d, \text{ and } k = c;$$

(ii) left and middle $\mathbb{C}$-associative with respect to span{1, i} if and only if

$$f = b,\ g = -a,\ h = d, \text{ and } k = -c;$$

An algebra $\mathfrak{A}$ with Table (T_m) is

(iii) left and middle $\mathbb{C}$-associative with respect to span{1, i} if and only if

$$f = b,\ g = -a,\ h = d, \text{ and } k = -c;$$

(iv) right and middle $\mathfrak{C}$-associative with respect to span$\{1, i\}$ if and only if

$$f = b,\ g = -a,\ h = -d, \text{ and } k = c;$$

An algebra $\mathfrak{A}$ with Table (T_ℓ) or Table (T_m) is

(v) fully-$\mathfrak{C}$-associative with respect to span$\{1, i\}$ if and only if

$$f = b,\ g = -a, \text{ and } c = d = h = k = 0;$$

(vi) associative if and only if

$$g = -a \text{ and } b = c = d = f = h = k = 0;$$

(vii) isomorphic to the quaternion algebra if and only if

$$g = -a > 0 \text{ and } b = c = d = f = h = k = 0.$$

Proof. See [4] and [5]. In the case of algebras satisfying (vi) or (vii), the basis change to $\{1, i, J/\sqrt{|a|}\ J, iJ/\sqrt{|a|}\}$ makes $J^2 = \pm 1$. Thus, there are really only two distinct associative $\mathfrak{C}$-associative algebras.

THEOREM 1.7. Unless the algebra $\mathfrak{A}$ with Table (T_ℓ) or Table (T_m) is associative, the only subalgebra with respect to which it satisfies any $\mathfrak{C}$-associative property is the subalgebra generated by 1 and i.

Proof. See the arguments in Theorem 6.8 in [4] or Theorem 6.2 in [5].

2. $\mathfrak{C}$-ASSOCIATIVE DIVISION ALGEBRAS

Recall that a division algebra is an algebra in which the equations $ax = b$ and $xa = b$ have unique solutions if $a \neq 0$. In the finite dimensional case this is equivalent to the absence of zero-divisors (i.e. products $ab = 0$ in which neither a nor b is 0). [9] The only associative real division algebras are the reals themselves, the complex numbers and the quaternions.

It is natural to ask which of the $\mathbb{C}$-associative algebras are division algebras. Commutative four-dimensional real algebras cannot be division algebras [16] and thus our search is restricted to the noncommutative case. The following theorem answers our question.

THEOREM 2.1. The algebra $\mathbb{A}$ with Table (T_ℓ) or Table (T_m) is a division algebra if and only if $A > 0$ and the polynomial

$$P(u) = (u^2 + Bu + A)^2 - u[(Cu + D)^2 + (Fu + G)^2]$$

has no positive roots, where

$A = bf - ag$ $\qquad$ $C = c + k$ $\qquad$ $F = d - h$

$B = g - a + ck - dh$ $\qquad$ $D = cg - bh + df - ak$ $\qquad$ $G = ah - bk - cf + dg$

Proof. See [4] and [5].

Sturm's Theorem yields conditions on the coefficients of a quartic that guarantee it will have no positive roots. (See [12].) Except in the following special cases, these conditions appear to be difficult to obtain explicitly:

The quaternion algebra: $P(u) = (u + 1)^4$

Fully-$\mathbb{C}$-associative algebras: $P(u) = [(u - a)^2 + b^2]^2$

Here $P(u)$ has no positive roots if and only if (i) $b \neq 0$ or else (ii) $b = 0$ and $a < 0$

Two-$\mathbb{C}$-associative algebras: $P(u) = [(u - a)^2 - (c^2 + d^2)u + b^2]^2$

Here $P(u)$ has no positive roots if and only if $(2a + c^2 + d^2)^2 < 4(a^2 + b^2)$; See Theorem 10 in [6].

We now turn to the question: what are the isomorphism classes of division algebras having $\mathbb{C}$-associative conditions?

In Theorems 2.2 and 2.3 we suppose $\mathbb{A}$ and $\mathbb{B}$ are algebras with tables of the form (T_ℓ) or (T_m) with respect to bases $\{1, i, J, iJ\}$ and $\{1, i', J', i'J'\}$. Parameters in the table for $\mathbb{B}$ will be denoted $a', b', \equiv, k'$.

THEOREM 2.2. Left $\mathbb{C}$-associative algebras $\mathfrak{A}$ and $\mathfrak{B}$ with tables of the form (T_ℓ) are isomorphic if and only if there exist real numbers $r \neq 0$ and θ such that

$$\begin{bmatrix} c \\ h \end{bmatrix} = rR_\theta \begin{bmatrix} c' \\ \pm h' \end{bmatrix}$$

$$\begin{bmatrix} d \\ k \end{bmatrix} = rR_\theta \begin{bmatrix} \pm d' \\ k' \end{bmatrix}$$

$$\begin{bmatrix} a+g \\ b-f \end{bmatrix} = r^2 R_{-2\theta} \begin{bmatrix} a'+g' \\ \pm(b'-f') \end{bmatrix}$$

$$\begin{bmatrix} a-g \\ b+f \end{bmatrix} = r^2 \begin{bmatrix} a'-g' \\ \pm(b'+f') \end{bmatrix}$$

Proof. See Theorem 3.5 in [4].

THEOREM 2.3. Middle but not fully $\mathbb{C}$-associative real division algebras $\mathfrak{A}$ and $\mathfrak{B}$ with tables of the form (T_m) are isomorphic if and only if there exist real numbers $r \neq 0$ and θ such that

$$\begin{bmatrix} c & d \\ h & k \end{bmatrix} = r \begin{bmatrix} c' & \pm d' \\ \pm h' & k' \end{bmatrix} R_\theta$$

$$\begin{bmatrix} a & b \\ f & g \end{bmatrix} = r^2 \begin{bmatrix} a' & \pm b' \\ \pm f' & g' \end{bmatrix}$$

Proof. See Theorem 3.3 in [5].

3. DERIVATIONS

DEFINITION. If $\mathfrak{E}$ is an algebra over a ring $\mathfrak{R}$, a *derivation* of $\mathfrak{E}$ is an $\mathfrak{R}$-linear transformation $\delta\colon \mathfrak{E} \varnothing \mathfrak{E}$ such that $\delta(xy) = x\delta(y) + \delta(x)y$, for all x, y in $\mathfrak{E}$. The set of derivations of $\mathfrak{E}$ forms a Lie algebra, Der $\mathfrak{E}$.

Benkart and Osborn [11] classified finite-dimensional real division algebras according to their derivation algebras. $\mathbb{C}$-associative algebras can also be classified according to their derivation algebras.

Commutative $\mathbb{C}$-associative algebras are given by the three tables (S_1), (S_2), and (T_c). The derivation algebras for the first two algebras are given in Theorem 11 in [6]. The derivation algebra for the middle $\mathbb{C}$-associative algebra is given in Theorem 4.2 in [5]. In summary:

(a) if $\mathfrak{A}$ has Table (S_1), then Der $\mathfrak{A}$ is the abelian two-dimensional real Lie algebra;

(b) otherwise Der $\mathfrak{A} = \{0\}$;

Theorems 12 and 13 in [6] and Theorem 4.1 in [4] describe the derivation algebras for all noncommutative four-dimensional left-$\mathbb{C}$-associative algebras (and hence right $\mathbb{C}$-associative algebras). In summary:

(i) When $\mathfrak{A}$ is fully $\mathbb{C}$-associative, so $J^2 = a + bi$:

- (a) if $b \neq 0$, then Der $\mathfrak{A} \cong \mathbb{R}$;
- (b) if $a = b = 0$, then Der $\mathfrak{A}$ is a nonabelian semidirect product of two abelian two-dimensional Lie algebras;
- (c) if $b = 0$ and $a > 0$, then Der $\mathfrak{A} \cong sl_2(\mathbb{R})$;
- (d) if $b = 0$ and $a < 0$, then Der $\mathfrak{A}$ is isomorphic to the vector product algebra $(\mathbb{R}^3, \infty)$.

(ii) When $\mathbb{A}$ is noncommutative and left and either right or middle $\mathbb{C}$-associative: Der $\mathbb{A} = \{0\}$.

(iii) When $\mathbb{A}$ is noncommutative and strictly left $\mathbb{C}$-associative: Der $\mathbb{A} = \{0\}$.

The situation for noncommutative strictly middle $\mathbb{C}$-associative algebras includes two other cases:

(iv) If $\mathbb{A}$ has Table (T_m) and $(c, d, h, k) = (0, 0, 0, 0)$ but $(a + g, b, f) \neq (0, 0, 0)$, then Der $\mathbb{A} \cong \mathbb{R}$.

(v) If $\mathbb{A}$ has Table (T_m) and $(c, d, h, k) \neq (0, 0, 0, 0)$, then Der $\mathbb{A} = \{0\}$.

4. GENERATORS FOR DIVISION ALGEBRAS

Yet another approach to classification counts the minimum number of elements in an algebra such that linear combinations of their products generate the entire algebra.

DEFINITIONS. The *nullity* of an algebra $\mathbb{A}$ is the minimum number of generators of $\mathbb{A}$. An algebra $\mathbb{A}$ with identity is *quadratic* if for each x in $\mathbb{A}$, $\{1, x, x^2\}$ is linearly dependent.

THEOREM 4.1 Any four-dimensional left $\mathbb{C}$-associative real division algebra that is not the quaternions has nullity one.
Proof. See Theorem 5.4 in [4].

THEOREM 4.2. Suppose $\mathbb{A}$ is a middle $\mathbb{C}$-associative division algebra with Table (T_m). Then following are equivalent

(i) $b = c = d = 0$;

(ii) $\mathbb{A}$ is quadratic;

(iii) $\mathbb{A}$ is a nullity-two algebra with identity;

(iv) $\mathbb{A}$ is power associative.

Proof. See Theorem 5.2 in [5].

There is a large class of algebras satisfying the conditions in Theorem 4.2. Note that if we take $b = c = d = h = k = 0$ and $a = -g = -1$ in Table (T_m), then the quartic in the division algebra condition is simply $P(u) = (u + 1)^4$. Hence, such algebras are always division algebras regardless of the choice of f. If $f = 0$, we obtain the quaternion algebra. Otherwise, we obtain a nonassociative algebra by Theorem 1.6(vi).

5. INTERSECTIONS WITH OTHER CLASSES OF DIVISION ALGEBRAS

Over the past dozen years other types of four-dimensional real division algebras have been discovered. In this section we summarize the relationship of the ℂ-associative algebras to two of these classes: rotational scaled quaternion algebras and fused algebras.

DEFINITION. [8] A *rotational scaled quaternion* algebra is a four-dimensional real algebra with basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with respect to which it has the table

(T_r)

	α_1	α_2	α_3	α_4
α_1	$q\alpha_1$	$r\alpha_2$	$u\alpha_3$	$u\alpha_4$
α_2	$s\alpha_2$	$t\alpha_1$	$v\alpha_4$	$-v\alpha_3$
α_3	$w\alpha_3$	$x\alpha_4$	$y\alpha_1$	$z\alpha_2$
α_4	$w\alpha_4$	$-x\alpha_3$	$-z\alpha_2$	$y\alpha_1$

THEOREM 5.1. A rotational scaled quaternion division algebra is left or right ℂ-associative if and only if it is the quaternion algebra.

Proof. See Theorem 6.1 in [4].

THEOREM 5.2. The algebra $\mathfrak{A}$ is a four-dimensional rotational scaled quaternion middle $\mathfrak{C}$-associative division algebra if and only if $\mathfrak{A}$ has a table of the form:

		e_1	e_2	e_3	e_4
	e_1	e_1	e_2	e_3	e_4
$(T_{r\&m})$	e_2	e_2	$-e_1$	e_4	$-e_3$
	e_3	e_3	$-e_4$	$-e_1$	ze_2
	e_4	e_4	e_3	$-ze_2$	$-e_1$

where $z > 0$.

Proof. See Theorem 6.1 in [5].

DEFINITION [7]. Suppose $\mathfrak{A} = (\mathbb{R}^2, \cdot)$ and $\mathfrak{B} = (\mathbb{R}^2, \infty)$ are two-dimensional real algebras with the following multiplication tables with respect to a basis $\{u, v\}$ for $\mathbb{R}^2$:

	$\cdot$	u	v
$\mathfrak{A}$	u	$a_{11}u + b_{11}v$	$a_{12}u + b_{12}v$
	v	$a_{21}u + b_{21}v$	$a_{22}u + b_{22}v$

	∞	u	v
$\mathfrak{B}$	u	$c_{11}u + d_{11}v$	$c_{12}u + d_{12}v$
	v	$c_{21}u + d_{21}v$	$c_{22}u + d_{22}v$

By analogy with Dickson's construction [9], we define multiplication on the direct sum $\mathfrak{A} \oplus \mathfrak{B}$ by

$$(\alpha, \int)(\copyright, \delta) = (\alpha \cdot \gamma - \beta \infty \delta, \alpha \cdot \delta + \beta \infty \gamma)$$

With this multiplication the direct sum $\mathfrak{A} \oplus \mathfrak{B}$ is called a *fused* algebra, and it is easy to write the multiplication table for it with respect to the ordered basis

$$\{(u, 0), (v, 0), (0, u), (0, v)\}.$$

Suppose $\mathfrak{A}$ is a fused division algebra, which also satisfies at least one $\mathfrak{C}$-associative condition. Then $\mathfrak{A}$ has an identity which must be its only idempotent. Thus span{(u, 0), (v, 0)} is a two-dimensional division algebra with identity, and hence is isomorphic to $\mathbb{C}$. It follows that by careful selection of u and v we can find a table for $\mathfrak{A}$ of the form:

(T_f^*)

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	$-e_1$	e_4	$-e_3$
e_3	e_3	$c_{12}e_3 + d_{12}e_4$	$-e_1$	$-c_{12}e_1 - d_{12}e_2$
e_4	e_4	$c_{22}e_3 + d_{22}e_4$	$-e_2$	$-c_{22}e_1 - d_{22}e_2$

From [7], this algebra is a division algebra if and only if

$$d_{12} < 0 \text{ and } (c_{12} - d_{22})^2 < - 4c_{22}d_{12}$$

The following results, proved in [4], describe the overlap between fused and right or left $\mathbb{C}$-associative algebras and classify these overlapping algebras. It turns out there are no strictly middle $\mathbb{C}$-associative fused division algebras. See Theorem 6.3 in [5].

THEOREM 5.3. A fused division algebra with table (T_f^*) is left $\mathbb{C}$-associative if and only if $d_{12} = -1$, $c_{12} = -d_{22}$, and $c_{22} = 1 + c_{12}{}^2$.

Proof. See Theorem 6.2 in [4].

THEOREM 5.4. A fused division algebra with table (T_f^*) is right $\mathbb{C}$-associative if and only if $d_{12} = -1$.

Proof. See Theorem 6.5 in [4].

The following corollaries are respectively corollaries 6.3, 6.4, 6.6, and 6.7 in [4].

COROLLARY 5.5. A fused left $\mathbb{C}$-associative division algebra $\mathfrak{A}$ with table (T_f^*) is right $\mathbb{C}$-associative; $\mathfrak{A}$ is fully $\mathbb{C}$-associative if and only if it is the quaternion algebra.

COROLLARY 5.6. Fused left $\mathbb{C}$-associative division algebras $\mathbb{A}$ and $\mathbb{B}$ with tables of the form (T_f^*) with parameters c_{12} and c_{12}', respectively, are isomorphic if and only if $c_{12} = \pm c_{12}'$.

COROLLARY 5.7. A fused right $\mathbb{C}$-associative division algebra $\mathbb{A}$ with table (T_f^*) is:

(i) left $\mathbb{C}$-associative if and only if $c_{12} = -d_{22}$, $d_{12} = -1$, and $c_{22} = 1 + c_{12}^2$;

(ii) middle $\mathbb{C}$-associative if and only if $c_{12} = d_{22}$, $d_{12} = -1$, and $c_{22} = 1$;

(iii) fully $\mathbb{C}$-associative if and only if $c_{12} = d_{22} = 0$, $d_{12} = -1$, and $c_{22} = 1$ (so $\mathbb{A}$ is the quaternion algebra).

COROLLARY 5.8. Fused right $\mathbb{C}$-associative division algebras $\mathbb{A}$ and $\mathbb{B}$ with tables of the form (T_f^*) are isomorphic if and only if

(i) $(c_{12}, d_{22}) = \pm(c_{12}', d_{22}')$ and $c_{22} = c_{22}'$ or

(ii) $(c_{12}, d_{22}) = (0, 0)$ and $c_{22} = 1/c_{22}'$.

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