

The Fifteen Theorem, and Generalizations

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December 18, 2001

1. Introduction. In 1993, Conway and Schneeberger announced the following remarkable result:

Theorem 1 (“The Fifteen Theorem”) *If a positive-definite quadratic form having integer matrix represents every positive integer up to 15 then it represents every positive integer.*

The original proof of this theorem was never published, perhaps because several of the cases involved rather intricate arguments. A sketch of this original proof was given by Schneeberger in [4].

The purpose of this paper is 1) to give a short and direct proof of the Fifteen Theorem, and 2) to give a summary of our recent results towards some surprising extensions of the Fifteen Theorem. We intend to provide more detailed accounts of these results in [1] and [2] respectively.

We note that our proof of the Fifteen Theorem is in spirit much the same as that of the original unpublished arguments of Conway and Schneeberger; however, we are able to avoid their intricate case-by-case analysis, thereby obtaining a significantly simplified proof. These simplified arguments are then applicable in more general situations, which we shall outline in the final section.

2. Preliminaries. The Fifteen Theorem deals with quadratic forms which are positive-definite and have integer matrix. As is well-known, there is a natural bijection between classes of such forms and lattices having integer inner products; precisely, a quadratic form f can be regarded as the inner product form for a corresponding lattice $L(f)$. Hence we shall oscillate freely between the language of forms and the language of lattices. For brevity, by a “form” we shall always mean a positive-definite quadratic form having integer matrix, and by a “lattice” we shall always mean a lattice having integer inner products.

A form (or its corresponding lattice) is said to be *universal* if it represents every positive integer. If a form f happens not to be universal, define the *truant* of f (or of its corresponding lattice $L(f)$) to be the smallest positive integer not represented by f .

Important in the proof of the Fifteen Theorem is the notion of “escalator lattice.” An *escalation* of a nonuniversal lattice L is defined to be any lattice which is generated by L and a vector whose norm is equal to the truant of L . An *escalator lattice* is a lattice which can be obtained as the result of a sequence of successive escalations of the zero-dimensional lattice.

3. Small-dimensional Escalators. The unique escalation of the zero-dimensional lattice is the lattice generated by a single vector of norm 1. This lattice corresponds to the form x^2 (or, in matrix form, $[1]$) which fails to represent the number 2. Hence an escalation of $[1]$ has inner product matrix of the form

$$\begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix}.$$

By the Cauchy-Schwartz inequality, $a^2 \leq 2$, so a equals either 0 or ± 1 . The choices $a = \pm 1$ lead to isometric lattices, so we obtain only two nonisometric two-dimensional escalators, namely those lattices having Minkowski-reduced Gram matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

If we escalate each of these two-dimensional escalators in the same manner, we find that we obtain exactly 9 new nonisometric escalator lattices, namely those having Minkowski-reduced Gram matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

Escalating now each of these nine three-dimensional escalators, we find exactly 207 non-isomorphic four-dimensional escalator lattices. All such lattices are of the form $[1] \oplus L$, and the 207 such lattices L are listed in Table 3. (We use the customary shorthand “ $D: a b c d e f$ ” to represent the three-dimensional lattice $\begin{bmatrix} a & f/2 & e/2 \\ f/2 & b & d/2 \\ e/2 & d/2 & c \end{bmatrix}$ of determinant D .)

When attempting to carry out the escalation process just once more, however, we find that many of the 207 four-dimensional lattices do not escalate (i.e., they are universal). For instance, one of the four-dimensional escalators turns out to be the lattice corresponding to the famous four squares form, $a^2 + b^2 + c^2 + d^2$, which is classically known to represent all integers. The question arises: how many of the four-dimensional escalators are universal?

4. Four-dimensional Escalators. In this section, we prove that in fact 201 of the 207 four-dimensional escalator lattices are universal; that is to say, only 6 of the four-dimensional escalators can be escalated once again.

The proof of universality of these 201 lattices proceeds as follows. In each such four-dimensional lattice L_4 , we locate a 3-dimensional sublattice L_3 which is known to represent some large set of integers. Typically, we simply choose L_3 to be unique in its genus; in that case, L_3 represents all integers that it represents locally (i.e., over each p -adic ring \mathbb{Z}_p). Armed with this knowledge of L_3 , we then show that the direct sum of L_3 with its orthogonal complement in L_4 represents all sufficiently large integers $n \geq N$. A check of representability of n for all $n < N$ finally reveals that L_4 is indeed universal.

To see this argument in practice, we consider in detail the escalations L_4 of the escalator lattice $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (labelled (4) in Table 1). The latter 3-dimensional lattice L_3 is unique in its genus, so a quick local calculation shows that it represents all positive integers not of the form $2^e(8k+7)$, where e is even. Let the orthogonal complement of L_3 in L_4 have Gram matrix $[m]$. We wish to show that $L_3 \oplus [m]$ represents all sufficiently large integers.

To this end, suppose L_4 is not universal, and let u be the first integer not represented by L_4 . Then, in particular, u is not represented by L_3 , so u must be of the form $2^e(8k+7)$. Moreover, u must be squarefree; for if $u = rt^2$ with $t > 1$, then $r = u/t^2$ is also not represented by L_4 , contradicting the minimality of u . Therefore $e = 0$, and we have $u \equiv 7 \pmod{8}$.

Now if $m \not\equiv 0, 3$ or $7 \pmod{8}$, then clearly $u - m$ is not of the form $2^e(8k+7)$. Similarly, if $m \equiv 3$ or $7 \pmod{8}$, then $u - 4m$ cannot be of the form $2^e(8k+7)$. Thus if $m \not\equiv 0 \pmod{8}$, and given that $u \geq 4m$, then either $u - m$ or $u - 4m$ is represented by L_3 ; that is, u is represented by $L_3 \oplus [m]$ (a sublattice of L_4) for $u \geq 4m$. An explicit calculation shows that m never exceeds 28, and a computer check verifies that every escalation L_4 of L_3 represents all integers less than $4 \times 28 = 112$. It follows that any escalator L_4 arising from L_3 , for which the value of m is not a multiple of 8, is universal.

Of course, the argument fails for those L_4 for which m is a multiple of 8. We call such an escalation ‘‘exceptional’’. Fortunately, such exceptional escalations are few and far between, and are easily handled. For instance, an explicit calculation shows that only two escalations of $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ are exceptional (while the other 24 are not); these exceptional cases are listed in Table 2.1. As is also indicated in the table, although these lattices did escape our initial attempt at proof, the universality of these four-dimensional lattices L_4 is still not any more difficult to prove; we simply change the sublattice L_3 from the escalator lattice $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ to the ones listed in the table, and apply the same argument!

It turns out that all of the 3-dimensional escalator lattices listed in Table 1, except for the one labeled (6), are unique in their genus, so the universality of their escalations can be proved by essentially identical arguments, with just a few exceptions. As for escalator (6), although not unique in its genus, it does represent all numbers locally represented by it except possibly those which are 7 or 10 (mod 12). Indeed, this escalator contains the lattice $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 8 \end{bmatrix}$, which is unique in its genus, and the lattices $\begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 2 \\ 2 & 2 & 8 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{bmatrix}$, which together form a genus; a local check shows that the first genus represents all numbers locally represented by escalator (6) which are not congruent to 2 or 3 (mod 4), while the second represents all such numbers not congruent to 1 (mod 3). The desired conclusion follows. (This fact has been independently proven by Kaplansky [3] using different methods.)

Knowing this, we may now proceed with essentially the same arguments on the escalations of $L_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$. The relevant portions of the proofs for all nonexceptional cases are

summarized in Table 1.

“Exceptional” cases arise only for escalators (4) (as we have already seen), (6), and (7). Two arise for escalator (4). Although four arise for escalator (6), two of them turn out to be nonexceptional escalations of (1) and (8) respectively, and hence have already been handled. Similarly, two arise for escalator (7), but one is a nonexceptional escalation of (9). Thus only five truly exceptional four-dimensional escalators remain, and these are listed in Table 2. In these five exceptional cases, other 3-dimensional sublattices unique in their genus are given for which essentially identical arguments work in proving universality. Again, all the relevant information is provided in Table 2.

5. Five-dimensional Escalators. As mentioned earlier, there are 6 four-dimensional escalators which escalate again; these are listed in the first column of Table 4. A rather large calculation shows that these 6 four-dimensional lattices escalate to an additional 1630 five-dimensional escalators! With a bit of fear we may ask again whether any of these five-dimensional escalators escalate.

Fortunately, the answer is no; all five-dimensional escalators are universal. The proof is much the same as the proof of universality of the four-dimensional escalators, but easier. We simply observe that, for the 6 four-dimensional nonuniversal escalators, all parts of the proof of universality outlined in the second paragraph of Section 4 go through—except for the final check. The final check then reveals that each of these 6 lattices represent every positive integer *except for one single number n* . Hence once a single vector of norm n is inserted in such a lattice, the lattice must automatically become universal. Therefore all five-dimensional escalators are universal. A list of the 6 nonuniversal four-dimensional lattices, together with the single numbers they fail to represent, is given in Table 4.

Since no five-dimensional escalator can be escalated, it follows that there are only finitely many escalator lattices: 1 of dimension zero, 1 of dimension one, 2 of dimension two, 9 of dimension three, 207 of dimension four, and 1630 of dimension five, for a total of 1850.

6. Remarks on the Fifteen Theorem. It is now obvious that

(i) *Any universal lattice L contains a universal sublattice of dimension at most five.*

For we can construct an escalator sequence $0 = L_0 \subseteq L_1 \subseteq \dots$ within L , and then from Sections 4 and 5, we see that either L_4 or (when defined) L_5 gives a universal escalator sublattice of L .

Our next remark includes the Fifteen Theorem.

(ii) *If a positive-definite quadratic form having integer matrix represents the nine critical numbers 1, 2, 3, 5, 6, 7, 10, 14, and 15, then it represents every positive integer. (Equivalently, the truant of any nonuniversal form must be one of these nine numbers.)*

This is because examination of the proof shows that only these numbers arise as truant of escalator lattices.

We note that Remark (ii) is the best possible statement of the Fifteen Theorem, in the following sense.

(iii) *If t is any one of the above critical numbers, then there is a quaternary diagonal form that fails to represent t , but represents every other positive integer.*

Nine such forms of minimal determinant are $[2, 2, 3, 4]$ with truant 1, $[1, 3, 3, 5]$ with truant 2, $[1, 1, 4, 6]$ with truant 3, $[1, 2, 6, 6]$ with truant 5, $[1, 1, 3, 7]$ with truant 6, $[1, 1, 1, 9]$ with truant 7, $[1, 2, 3, 11]$ with truant 10, $[1, 1, 2, 15]$ with truant 14, and $[1, 2, 5, 5]$ with truant 15.

However, there is another slight strengthening of the Fifteen Theorem, which shows that the number 15 is rather special:

(iv) *If a positive-definite quadratic form having integer matrix represents every number below 15, then it represents every number above 15.*

This is because there are only four escalator lattices having truant 15, and as was shown in Section 5, each of these four escalators represents every number greater than 15.

Fifteen is the smallest number for which Remark (iv) holds. In fact:

(v) *There are forms which miss infinitely many integers starting from any of the eight critical numbers not equal to 15.*

Indeed, in each case one may simply take an appropriate escalator lattice of dimension one, two, or three.

(vi) *There are exactly 204 universal quaternary forms.*

An upper bound for the discriminant of such a form is easily determined; a systematic use of the Fifteen Theorem then yields the desired result. We note that the enumeration of universal quaternary forms was previously carried out in the well-known work of Willerding [5], who “showed” that there are exactly 178 universal quaternary forms. However, a comparison with our tables shows that Willerding actually missed 36 universal forms, listed one universal form twice, and listed 9 non-universal forms! A list of all 204 quaternary universal forms is given in Table 5; the three entries not appearing among the list of escalators in Table 3 have been italicized.

7. Extensions of the Fifteen Theorem. Combining the techniques exhibited here with analytic estimates for the growth of Fourier coefficients of modular forms, we may obtain a series of remarkable generalizations of the Fifteen Theorem, which show that the Conway-Schneeberger Fifteen Theorem is far from being just an isolated coincidence!

In [2], we obtain the following general finiteness result:

Theorem 2 *Let S be any subset of the nonnegative integers. Then there is a unique subset $T \subseteq S$ satisfying the following properties:*

(i) *A positive-definite integer-matrix quadratic form represents S if and only if it repre-*

sents T ;

- (ii) if T' also satisfies (i), then $T \subseteq T'$; and
- (iii) T is finite.

In other words, for any subset S of the natural numbers, there exists a unique, minimal finite subset T of S , such that a quadratic form represents S if and only if it represents T .

Interestingly, Theorem 2 is not effective in general. But for many choices of S that that have been of much classical interest in number theory, we have been able to effectively determine the corresponding set T ; namely, we have

- If S is the set of all natural numbers, then $T = \{1, 2, 3, 5, 6, 7, 10, 14, 15\}$. (This is the Conway-Schneeberger Fifteen Theorem.)
- If S is the set of odd natural numbers, then $T = \{1, 3, 5, 7, 11, 15, 33\}$.
- If S is the set of prime numbers, then $T = \{\text{primes up to } 47\} \cup \{67, 73\}$. (this is my favorite :))

For several additional examples of correspondences $S \leftrightarrow T$, see [2].

We note, finally, that we have examples of specific sets S for which T seems essentially indeterminable without significant progress on the Ramanujan conjecture for weight $3/2$ modular forms; thus the general effectivization of Theorem 2 is yet another concrete number-theoretic motivation for the resolution of Ramanujan's deep conjectures in the subject of modular forms.

Acknowledgments. I thank Professor Conway for many enlightening discussions, and for helpful comments on an early draft of this paper. I am grateful to Prof. Bumcrot and the entire Mathematics Department at Hofstra University for a wonderful time at the retirement conference in honor of Prof. Bumcrot, May 11, 2001.

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Three-dimensional escalator lattice	Truant	Represents nos. not of the form	If m	Subtract	Check up to
(1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	7	$2^e u_7$	$\not\equiv 0 \pmod{8}$ $\equiv 0 \pmod{8}$	m or $4m$ does not arise	112 -
(2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	14	$2^d u_7$	$\not\equiv 0 \pmod{16}$ $\equiv 0 \pmod{16}$	m or $4m$ does not arise	224 -
(3) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	6	$3^d u_-$	$\not\equiv 0 \pmod{9}$ $\equiv 0 \pmod{9}$	$m, 4m,$ or $16m$ does not arise	864 -
(4) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	7	$2^e u_7$	$\not\equiv 0 \pmod{8}$ $\equiv 0 \pmod{8}$	m or $4m$ [See Table 2]	112 -
(5) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	10	$2^d u_5$	$\not\equiv 0 \pmod{16}$ $\equiv 0 \pmod{16}$	m or $4m$ does not arise	1440 -
(6) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 4 \end{bmatrix}$	7	$7^d u_-$ or $7, 10 \pmod{12}$	$\not\equiv 0, 3, 9 \pmod{12}$ & $\not\equiv 0 \pmod{49}$ $\equiv 0 \pmod{49}$ $\equiv 0, 3, 9 \pmod{12}$	$m, 4m,$ or $9m$ does not arise [See Table 2]	3087 - -
(7) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$	14	$2^d u_7$	$\not\equiv 0 \pmod{16}$ $\equiv 0 \pmod{8}$	m or $4m$ [See Table 2]	224 -
(8) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 5 \end{bmatrix}$	7	$2^e u_7$	$\not\equiv 0 \pmod{8}$ $\equiv 0 \pmod{8}$	m or $4m$ does not arise	252 -
(9) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$	10	$5^d u_-$	$\not\equiv 0 \pmod{25}$ $\equiv 0 \pmod{25}$	m or $4m$ does not arise	4000 -

Table 1. Proof of universality of four-dimensional escalators (nonexceptional cases)

<u>“Exceptional” Lattice</u>	<u>New unique in genus sublattice</u>	<u>Unrepresented numbers</u>	<u>m</u>	<u>Subtract</u>	<u>Check up to</u>
$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 2 & 1 & 1 & 7 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$	$5^d u_+$	40	m or $4m$	160
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 7 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 7 \end{bmatrix}$	$2^e u_1, 2^e u_5,$ $2^d u_3, 2^d u_7, 3^d u_+$	1	m	14
$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 4 & 3 \\ 1 & 0 & 3 & 7 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{bmatrix}$	$2^d u_7$	1	$m, 4m,$ or $9m$	9
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 7 \end{bmatrix}^*$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 10 \end{bmatrix}$	$2^d u_7$	90	m or $4m$	504
$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 1 & 0 & 2 & 14 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 13 \end{bmatrix}$	$2^d u_5, 2^e u_3$	2	m or $4m$	8

Table 2. Proof of universality of four-dimensional escalators (exceptional cases)

1: 1 1 1 0 0 0	16: 2 3 3 2 0 0	30: 2 4 4 2 0 0	49: 2 3 9 2 2 0	72: 2 5 8 4 0 0
2: 1 1 2 0 0 0	17: 1 2 9 2 0 0	31: 2 3 6 2 2 0	49: 2 4 7 0 0 2	74: 2 4 10 2 2 0
3: 1 1 3 0 0 0	17: 1 3 6 2 0 0	31: 2 4 5 0 2 2	49: 2 5 6 0 2 2	76: 2 4 10 0 2 0
3: 1 2 2 2 0 0	17: 2 3 4 0 2 2	32: 2 4 4 0 0 0	50: 2 4 7 2 2 0	77: 2 5 9 4 2 0
4: 1 1 4 0 0 0	18: 1 2 9 0 0 0	32: 2 4 5 4 0 0	<i>50: 2 5 5 0 0 0</i>	78: 2 4 10 2 0 0
4: 1 2 2 0 0 0	18: 1 3 6 0 0 0	33: 2 3 6 0 2 0	51: 2 3 9 0 2 0	78: 2 5 8 2 0 0
4: 2 2 2 2 2 0	18: 2 2 5 2 0 0	<i>33: 2 4 5 2 0 2</i>	52: 2 3 9 2 0 0	80: 2 4 10 0 0 0
5: 1 1 5 0 0 0	18: 2 3 3 0 0 0	34: 2 3 6 2 0 0	52: 2 5 6 2 0 2	80: 2 4 11 4 0 0
5: 1 2 3 2 0 0	18: 2 3 4 2 0 2	34: 2 4 5 2 2 0	52: 2 5 6 4 0 0	80: 2 5 8 0 0 0
6: 1 1 6 0 0 0	19: 1 2 10 2 0 0	34: 2 4 6 4 0 2	53: 2 5 6 2 2 0	82: 2 4 11 2 2 0
6: 1 2 3 0 0 0	19: 2 3 4 2 2 0	35: 2 4 5 0 0 2	54: 2 3 9 0 0 0	82: 2 5 9 4 0 0
6: 2 2 2 2 0 0	20: 1 2 10 0 0 0	36: 2 3 6 0 0 0	54: 2 4 7 2 0 0	<i>83: 2 5 9 2 2 0</i>
7: 1 1 7 0 0 0	20: 2 2 5 0 0 0	36: 2 4 5 0 2 0	54: 2 5 6 0 0 2	85: 2 5 9 0 2 0
7: 1 2 4 2 0 0	20: 2 2 6 2 2 0	36: 2 4 6 4 2 0	54: 2 5 7 4 2 2	86: 2 4 11 2 0 0
7: 2 2 3 2 0 2	20: 2 4 4 4 2 0	36: 2 5 5 4 2 2	55: 2 3 10 2 2 0	87: 2 5 10 4 2 0
8: 1 2 4 0 0 0	<i>21: 2 3 4 0 2 0</i>	37: 2 5 5 4 2 0	55: 2 5 6 0 2 0	88: 2 4 11 0 0 0
8: 1 3 3 2 0 0	22: 1 2 11 0 0 0	38: 2 4 5 2 0 0	55: 2 5 7 4 0 2	88: 2 4 12 4 0 0
8: 2 2 2 0 0 0	22: 2 2 6 2 0 0	38: 2 4 6 0 2 2	56: 2 4 7 0 0 0	88: 2 5 9 2 0 0
8: 2 2 3 2 2 0	22: 2 3 4 2 0 0	39: 2 3 7 0 2 0	56: 2 4 8 4 0 0	90: 2 4 12 2 2 0
9: 1 2 5 2 0 0	22: 2 3 5 0 2 2	40: 2 3 7 2 0 0	57: 2 3 10 0 2 0	90: 2 5 9 0 0 0
9: 1 3 3 0 0 0	23: 1 2 12 2 0 0	40: 2 4 5 0 0 0	58: 2 3 10 2 0 0	92: 2 4 13 4 2 0
9: 2 2 3 0 0 2	23: 2 3 5 2 0 2	40: 2 4 6 2 0 2	58: 2 4 8 2 2 0	92: 2 5 10 4 0 0
10: 1 2 5 0 0 0	24: 1 2 12 0 0 0	40: 2 4 6 4 0 0	58: 2 5 6 2 0 0	93: 2 5 10 2 2 0
10: 2 2 3 2 0 0	24: 2 2 6 0 0 0	41: 2 4 7 4 0 2	58: 2 5 7 0 2 2	94: 2 4 12 2 0 0
10: 2 2 4 2 0 2	24: 2 2 7 2 2 0	42: 2 3 7 0 0 0	60: 2 3 10 0 0 0	95: 2 5 10 0 2 0
11: 1 2 6 2 0 0	24: 2 3 4 0 0 0	42: 2 4 6 0 0 2	60: 2 4 9 4 2 0	96: 2 4 12 0 0 0
11: 1 3 4 2 0 0	24: 2 4 4 0 2 2	42: 2 4 6 2 2 0	60: 2 5 6 0 0 0	96: 2 4 13 4 0 0
12: 1 2 6 0 0 0	24: 2 4 4 4 0 0	42: 2 5 5 4 0 0	61: 2 5 7 2 0 2	98: 2 4 13 2 2 0
12: 1 3 4 0 0 0	25: 1 2 13 2 0 0	43: 2 3 8 2 2 0	62: 2 4 8 2 0 0	98: 2 5 10 2 0 0
12: 2 2 3 0 0 0	25: 2 3 5 2 2 0	<i>43: 2 5 5 2 0 2</i>	62: 2 5 7 4 0 0	100: 2 4 13 0 2 0
12: 2 2 4 0 0 2	26: 1 2 13 0 0 0	44: 2 4 6 0 2 0	63: 2 5 7 0 0 2	100: 2 4 14 4 2 0
13: 2 2 5 2 0 2	26: 2 2 7 2 0 0	45: 2 4 7 0 2 2	63: 2 5 7 2 2 0	100: 2 5 10 0 0 0
13: 2 3 3 2 2 0	26: 2 4 4 2 2 0	45: 2 5 5 0 2 0	64: 2 4 8 0 0 0	102: 2 4 13 2 0 0
14: 1 2 7 0 0 0	27: 1 2 14 2 0 0	45: 2 5 6 4 2 2	66: 2 4 9 2 2 0	104: 2 4 13 0 0 0
14: 1 3 5 2 0 0	27: 2 3 5 0 2 0	46: 2 3 8 2 0 0	<i>67: 2 5 8 4 2 0</i>	104: 2 4 14 4 0 0
14: 2 2 4 2 0 0	27: 2 4 5 4 0 2	46: 2 4 6 2 0 0	68: 2 4 9 0 2 0	106: 2 4 14 2 2 0
15: 1 2 8 2 0 0	28: 1 2 14 0 0 0	46: 2 5 6 4 0 2	68: 2 4 10 4 2 0	108: 2 4 14 0 2 0
15: 1 3 5 0 0 0	28: 2 2 7 0 0 0	47: 2 4 7 2 0 2	68: 2 5 7 2 0 0	110: 2 4 14 2 0 0
15: 2 2 5 0 0 2	28: 2 3 5 2 0 0	47: 2 5 6 4 2 0	70: 2 4 9 2 0 0	112: 2 4 14 0 0 0
15: 2 3 3 0 2 0	28: 2 4 4 0 2 0	48: 2 3 8 0 0 0	70: 2 5 7 0 0 0	
16: 1 2 8 0 0 0	28: 2 4 5 4 2 0	48: 2 4 6 0 0 0	72: 2 4 9 0 0 0	
16: 2 2 4 0 0 0	30: 2 3 5 0 0 0	48: 2 5 5 2 0 0	72: 2 4 10 4 0 0	

Table 3. Ternary forms L such that $[1] \oplus L$ is an escalator.
(Entries not appearing in Table 5 are italicized.)

Nonuniversal four-dimensional escalator Unique number not represented

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix} \qquad 10$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad 10$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad 15$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \qquad 15$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 0 & 1 & 2 & 8 \end{bmatrix} \qquad 15$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 5 & 1 \\ 0 & 1 & 1 & 9 \end{bmatrix} \qquad 15$$

Table 4. Nonuniversal four-dimensional escalator lattices

1: 1 1 1 0 0 0	16: 1 2 8 0 0 0	28: 2 3 5 2 0 0	48: 2 3 8 0 0 0	72: 2 4 9 0 0 0
2: 1 1 2 0 0 0	16: 2 2 4 0 0 0	28: 2 4 4 0 2 0	48: 2 4 6 0 0 0	72: 2 4 10 4 0 0
3: 1 1 3 0 0 0	16: 2 3 3 2 0 0	28: 2 4 5 4 2 0	48: 2 5 5 2 0 0	72: 2 5 8 4 0 0
3: 1 2 2 2 0 0	17: 1 2 9 2 0 0	30: 2 3 5 0 0 0	49: 2 3 9 2 2 0	74: 2 4 10 2 2 0
4: 1 1 4 0 0 0	17: 1 3 6 2 0 0	30: 2 4 4 2 0 0	49: 2 4 7 0 0 2	76: 2 4 10 0 2 0
4: 1 2 2 0 0 0	17: 2 3 4 0 2 2	31: 2 3 6 2 2 0	49: 2 5 6 0 2 2	77: 2 5 9 4 2 0
4: 2 2 2 2 2 0	18: 1 2 9 0 0 0	31: 2 4 5 0 2 2	50: 2 4 7 2 2 0	78: 2 4 10 2 0 0
5: 1 1 5 0 0 0	18: 1 3 6 0 0 0	32: 2 4 4 0 0 0	51: 2 3 9 0 2 0	78: 2 5 8 2 0 0
5: 1 2 3 2 0 0	18: 2 2 5 2 0 0	32: 2 4 5 4 0 0	52: 2 3 9 2 0 0	80: 2 4 10 0 0 0
6: 1 1 6 0 0 0	18: 2 3 3 0 0 0	33: 2 3 6 0 2 0	52: 2 5 6 2 0 2	80: 2 4 11 4 0 0
6: 1 2 3 0 0 0	18: 2 3 4 2 0 2	34: 2 3 6 2 0 0	52: 2 5 6 4 0 0	80: 2 5 8 0 0 0
6: 2 2 2 2 0 0	19: 1 2 10 2 0 0	34: 2 4 5 2 2 0	53: 2 5 6 2 2 0	82: 2 4 11 2 2 0
7: 1 1 7 0 0 0	19: 2 3 4 2 2 0	34: 2 4 6 4 0 2	54: 2 3 9 0 0 0	82: 2 5 9 4 0 0
7: 1 2 4 2 0 0	20: 1 2 10 0 0 0	35: 2 4 5 0 0 2	54: 2 4 7 2 0 0	85: 2 5 9 0 2 0
7: 2 2 3 2 0 2	20: 2 2 5 0 0 0	36: 2 3 6 0 0 0	54: 2 5 6 0 0 2	86: 2 4 11 2 0 0
8: 1 2 4 0 0 0	20: 2 2 6 2 2 0	36: 2 4 5 0 2 0	54: 2 5 7 4 2 2	87: 2 5 10 4 2 0
8: 1 3 3 2 0 0	<i>20: 2 3 4 0 0 2</i>	36: 2 4 6 4 2 0	55: 2 3 10 2 2 0	88: 2 4 11 0 0 0
8: 2 2 2 0 0 0	20: 2 4 4 4 2 0	36: 2 5 5 4 2 2	55: 2 5 6 0 2 0	88: 2 4 12 4 0 0
8: 2 2 3 2 2 0	22: 1 2 11 0 0 0	37: 2 5 5 4 2 0	55: 2 5 7 4 0 2	88: 2 5 9 2 0 0
9: 1 2 5 2 0 0	22: 2 2 6 2 0 0	38: 2 4 5 2 0 0	56: 2 4 7 0 0 0	90: 2 4 12 2 2 0
9: 1 3 3 0 0 0	22: 2 3 4 2 0 0	38: 2 4 6 0 2 2	56: 2 4 8 4 0 0	90: 2 5 9 0 0 0
9: 2 2 3 0 0 2	22: 2 3 5 0 2 2	39: 2 3 7 0 2 0	57: 2 3 10 0 2 0	92: 2 4 13 4 2 0
10: 1 2 5 0 0 0	23: 1 2 12 2 0 0	40: 2 3 7 2 0 0	58: 2 3 10 2 0 0	92: 2 5 10 4 0 0
10: 2 2 3 2 0 0	23: 2 3 5 2 0 2	40: 2 4 5 0 0 0	58: 2 4 8 2 2 0	93: 2 5 10 2 2 0
10: 2 2 4 2 0 2	24: 1 2 12 0 0 0	40: 2 4 6 2 0 2	58: 2 5 6 2 0 0	94: 2 4 12 2 0 0
11: 1 2 6 2 0 0	24: 2 2 6 0 0 0	40: 2 4 6 4 0 0	58: 2 5 7 0 2 2	95: 2 5 10 0 2 0
11: 1 3 4 2 0 0	24: 2 2 7 2 2 0	41: 2 4 7 4 0 2	60: 2 3 10 0 0 0	96: 2 4 12 0 0 0
12: 1 2 6 0 0 0	24: 2 3 4 0 0 0	42: 2 3 7 0 0 0	60: 2 4 9 4 2 0	96: 2 4 13 4 0 0
12: 1 3 4 0 0 0	24: 2 4 4 0 2 2	42: 2 4 6 0 0 2	60: 2 5 6 0 0 0	98: 2 4 13 2 2 0
12: 2 2 3 0 0 0	24: 2 4 4 4 0 0	42: 2 4 6 2 2 0	61: 2 5 7 2 0 2	98: 2 5 10 2 0 0
12: 2 2 4 0 0 2	25: 1 2 13 2 0 0	42: 2 5 5 4 0 0	62: 2 4 8 2 0 0	100: 2 4 13 0 2 0
<i>12: 2 3 3 0 2 2</i>	<i>25: 2 3 5 0 0 2</i>	43: 2 3 8 2 2 0	62: 2 5 7 4 0 0	100: 2 4 14 4 2 0
13: 2 2 5 2 0 2	25: 2 3 5 2 2 0	44: 2 4 6 0 2 0	63: 2 5 7 0 0 2	100: 2 5 10 0 0 0
13: 2 3 3 2 2 0	26: 1 2 13 0 0 0	45: 2 4 7 0 2 2	63: 2 5 7 2 2 0	102: 2 4 13 2 0 0
14: 1 2 7 0 0 0	26: 2 2 7 2 0 0	45: 2 5 5 0 2 0	64: 2 4 8 0 0 0	104: 2 4 13 0 0 0
14: 1 3 5 2 0 0	26: 2 4 4 2 2 0	45: 2 5 6 4 2 2	66: 2 4 9 2 2 0	104: 2 4 14 4 0 0
14: 2 2 4 2 0 0	27: 1 2 14 2 0 0	46: 2 3 8 2 0 0	68: 2 4 9 0 2 0	106: 2 4 14 2 2 0
15: 1 2 8 2 0 0	27: 2 3 5 0 2 0	46: 2 4 6 2 0 0	68: 2 4 10 4 2 0	108: 2 4 14 0 2 0
15: 1 3 5 0 0 0	27: 2 4 5 4 0 2	46: 2 5 6 4 0 2	68: 2 5 7 2 0 0	110: 2 4 14 2 0 0
15: 2 2 5 0 0 2	28: 1 2 14 0 0 0	47: 2 4 7 2 0 2	70: 2 4 9 2 0 0	112: 2 4 14 0 0 0
15: 2 3 3 0 2 0	28: 2 2 7 0 0 0	47: 2 5 6 4 2 0	70: 2 5 7 0 0 0	

Table 5. Ternary forms L such that $[1] \oplus L$ is universal.
(Entries not appearing in Table 3 are italicized.)