

Groups as unions of subgroups

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1 Introduction

Group theory has always interested me, and when I was a graduate student, a problem of mine appeared in the American Mathematical Monthly, which some years later appeared also on the national William Lowell Putnam Examination. As it turned out, one of the solvers of my problem when it appeared in the American Mathematical Monthly was none other than Bob Bumcrot, and so I thought it would be particularly appropriate for this Festschrift in honor of Bob to discuss some beautiful extensions of this old problem, and in particular, to present some of my own very recent results in this direction.

2 Groups as unions of n proper subgroups

The problem mentioned above was as follows:

Question: When is a group the union of two of its proper subgroups?

As is now well-known, the answer is that a group can *never* be the union of two proper subgroups.

Answer: Never!

The next question then that immediately comes to mind is: when is a group the union of three proper subgroups?

A complete answer to this question was given by Scorza [6], who proved:

Theorem 1 (Scorza) *A group is the union of three proper subgroups if and only if it has a quotient isomorphic to $C_2 \times C_2$.*

One direction of the theorem is of course easy to see: $C_2 \times C_2$ is the union of its three subgroups H_1, H_2, H_3 of index 2, and if one has a surjective homomorphism $\phi : G \rightarrow C_2 \times C_2$, then G is the union of its proper subgroups $\phi^{-1}(H_1), \phi^{-1}(H_2), \phi^{-1}(H_3)$.

Scorza's result was so elegant that people thought it ought to be generalized to numbers higher than three. In 1994, Cohn [3] performed a detailed analysis and obtained some very interesting analogues for four, five and six. Following his definition, let us write $\sigma(G) = n$ whenever G is the union of n proper subgroups, but is not the union of any smaller number of proper subgroups. Thus, for instance, Scorza's result asserts that $\sigma(G) = 3$ if and only if G has a quotient isomorphic to $C_2 \times C_2$. Cohn's extensions for $\sigma(G) = 4, 5, \text{ and } 6$ are as follows:

Theorem 2 (Cohn) *Let G be a group. Then*

- (a) $\sigma(G) = 4$ if and only if G has a quotient isomorphic to S_3 or $C_3 \times C_3$.
- (b) $\sigma(G) = 5$ if and only if G has a quotient isomorphic to the alternating group A_4 .
- (c) $\sigma(G) = 6$ if and only if G has a quotient isomorphic to $D_5, C_5 \times C_5$, or W , where W is the group of order 20 defined by $a^5 = b^4 = \{e\}, ba = a^2b$.

The next case, $\sigma(G) = 7$, gets considerably more unwieldy, but in the end, the answer is amusing and rather unexpected. Cohn conjectured, and in 1997 Tomkinson [5] proved, that

Theorem 3 (Tomkinson) *There is no group G such that $\sigma(G) = 7$.*

The proof of Tomkinson's theorem was very complicated and involved indeed. Thus the idea of classifying groups that are the union of n proper subgroups for even higher n would no doubt be a formidable task, since there appears to be no consistent pattern for small n that suggests a general answer.

However, based on all the cases above, we make the following conjecture.

Conjecture 4 *For any positive integer n , there exists a minimal, finite set $S(n)$ of groups such that $\sigma(G) = n$ if and only if G has a quotient isomorphic to some group $K \in S(n)$.*

For example, $S(1) = S(2) = S(7) = \phi$, while $S(3) = \{C_2 \times C_2\}$, $S(4) = \{S_3, C_3 \times C_3\}$, $S(5) = \{A_4\}$, and $S(6) = \{D_5, C_5 \times C_5, W\}$.

3 Groups as unions of proper normal subgroups

As we mentioned in the previous section, the situation gets increasingly complicated as we require a group G to be the union of larger numbers of subgroups. This indicated to me that perhaps we might not be asking quite the right question in generalizing Scorza's result.

Notice that Scorza's theorem implies that

Theorem 5 *A group that is the union of three proper subgroups is also the union of three proper normal subgroups.*

Proof: By Scorza's theorem, there is a surjective homomorphism $\phi : G \rightarrow C_2 \times C_2$. Now $C_2 \times C_2$ is the union of three proper normal subgroups H_1, H_2, H_3 . Therefore, G is the union of the three proper normal subgroups $\phi^{-1}(H_1), \phi^{-1}(H_2), \phi^{-1}(H_3)$. \square

Thus we may add the word *normal* to Scorza's result, and it remains true. This suggests that we might ask: when is a group the union of proper normal subgroups?

We may call a group that is the union of its proper normal subgroups *anti-simple*, because such a group is quite the opposite of simple. Indeed, in an anti-simple group G , not only do there exist non-trivial proper normal subgroups, but every element of G is contained in one. The classification of finite simple groups was a huge collaborative effort which took over 20 years. Luckily, the classification of anti-simple groups did not take us quite so long. Moreover, the answer to our question turns out to be remarkably similar to Theorem 1:

Theorem 6 *A finite group G is the union of its proper normal subgroups if and only if it has a quotient isomorphic to $C_p \times C_p$ for some prime p .*

One direction is again easy to see: $C_p \times C_p$ is the union of its normal subgroups H_1, H_2, \dots, H_{p+1} of index p , and if $\phi : G \rightarrow C_p \times C_p$ is a surjective homomorphism, then G is the union of the proper normal subgroups $\phi^{-1}(H_1), \phi^{-1}(H_2), \dots, \phi^{-1}(H_{p+1})$.

The reverse direction is a good deal more involved, and hence we shall omit the proof here. (Readers may refer to my paper [2] for details.)

Theorem 6 above has some interesting offshoots. First, notice that, in Theorem 6, we did not mention how many normal subgroups we would like G to be the union of, and one may ask what is the smallest number of such normal subgroups required for an anti-simple group. Before addressing this question, we make the following definition.

Definition 7 *Let G be anti-simple, and let $\eta(G)$ denote the minimal number of proper normal subgroups that G is the union of.*

Our next theorem describes precisely the possible values of $\eta(G)$.

Theorem 8 *If G is anti-simple, then $\eta(G) = p + 1$, where p is the smallest prime such that G has a quotient isomorphic to $C_p \times C_p$.*

Next, we note that our main theorem, Theorem 6, applies also to infinite groups G , provided we require our group G to be the union of *finitely many* normal subgroups. If we drop this finiteness condition, however, then the theorem does not hold anymore. For example, for each $i = 1, 2, 3, \dots$, let G_i be a nonabelian simple group, and define $G = \bigoplus_{i=1}^{\infty} G_i$. For each $j = 1, 2, 3, \dots$, let H_j denote the (normal) subgroup $\bigoplus_{i=1}^j G_i$ of G . Then clearly $G = \bigcup_{j=1}^{\infty} H_j$, and hence is anti-simple; however, it may be seen that the commutator of G must be G itself, and so G cannot have a quotient isomorphic to $C_p \times C_p$, or any non-trivial abelian quotient. Therefore, in closing, we invite the reader to ponder the question:

Question: What is the classification of anti-simple *infinite* groups?

An elegant solution to the above problem would be a most welcome final touch to the complete classification of anti-simple groups.

References

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