Graphs and Pigeonholes

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Sometimes the pigeonhole principle (PHP) is used to prove a result in graph theory. Theorem 0 is a famous, well-known example.

Theorem 0 (Ramsey's Theorem) If the edges of a K_6 are colored with red and blue then there is a K_3 subgraph which is either all red or all blue. (see page 248, [1])

There are many other elegant applications of the pigeonhole principle (e.g., see [1], [2], [3] and [4]). Here are some novel graph theory interpretations of a few of these PHP results.

Theorem 1 Given any sequence of n complete graphs $G_1, G_2, G_3, ..., G_n$ (not necessarily isomorphic) there exist a consecutive subsequence $G_{k+1}, G_{k+2}, G_{k+3}, ..., G_m$ and an integer c ($1 \leq c$), such that, the join of this subsequence is isomorphic to the join of c copies of K_n. (see page 125, [2])

Theorem 2 Given the sequence of complete graphs $K_1, K_2, K_3, ..., K_{2n}$ then in any subsequence (not necessarily consecutive) of (n + 1) of these graphs there exist two complete graphs K_i and K_j within that subsequence and an integer c ($c > 1$), such that, K_i is the join of c copies of K_j. (see page 123, [2])

Theorem 3 Given the sequence of m (m odd) full complete binary trees $T_1, T_2, T_3, ..., T_m$ with T_i of depth (i - 1), there exist an integer n ($1 \leq n \leq m$), such that, $m \mid \text{card}(V(T_n))$. (see page 247, [3])

Theorem 4 Given any sequence of $m = n^2 + 1$ non-isomorphic paths $G_1, G_2, G_3, \ldots, G_m$ there is a (not necessarily consecutive) subsequence of paths $G'_1, G'_2, G'_3, \ldots, G'_{n+1}$, of size $(n + 1)$ such that, either G'_i is a subgraph of G'_{i+1} for all $1 \leq i \leq n$ or G'_i is a supergraph of G'_{i+1} for all $1 \leq i \leq n$. (see page 247, [1])

Theorem 5 Given any sequence of nontrivial path graphs $G_1, G_2, G_3, \ldots, G_m$ (not necessarily isomorphic) with lengths at most n and another sequence of nontrivial path graphs $H_1, H_2, H_3, \ldots, H_n$ (not necessarily isomorphic) with lengths at most m, $\exists$ p, q, r, s such that, the consecutive subsequence $G_p, G_{p+1}, G_{p+2}, \ldots, G_q$ and the consecutive subsequence $H_r, H_{r+1}, H_{r+2}, \ldots., H_s$, are such that, the concatenation of the first sequence $G_p \oplus G_{p+1} \oplus G_{p+2} \oplus \ldots \oplus G_q$ is isomorphic to the concatenation of the second sequence $H_r \oplus H_{r+1} \oplus H_{r+2} \oplus \ldots \oplus H_s$. (see page 190, [4])

Theorem 6 Let C_n ($n \geq 3$) be a cycle graph and let G be any proper subgraph of C_n with n vertices. If the number of edges in G exceeds $(k - 1) n / k$ then $\exists$ a path graph P_{k+1} which is a subgraph of G with k edges. (see page 192, [4])

Corrolary 7 Let C_n ($n \geq 3$) be a bi-chromatic cycle graph with $r > 0$ red egdes and $b > 0$ blue egdes ($r + b = n$) and let G be any proper subgraph of C_n with n vertices and g_r red edges and g_b blue edges ($g_r + g_b \leq n - 1$).
If $g_r > (k_r - 1) n / k_r$ then $\exists$ a path graph P_{kr+1}, a subgraph of G with k_r red edges and
if $g_b > (k_b - 1) n / k_b$ then $\exists$ a path graph P_{kb+1}, a subgraph of G with k_b blue edges.

Theorem 8 If $M_{n,n}$ is an n by n mesh graph on n^2 vertices with vertices labeled arbitrarily with values $1, 2, 3, \ldots, n^2$, and each edge uv weighted $|u - v|$, then $\exists$ an edge with weight at least n . (see page 194, [4])

References

[1] Rosen, K.H., Discrete Mathematics and Its Applications, 4th ed., WCB - McGraw Hill, 1999.

[2] Aigner, M., Ziegler, G.M., Proofs from THE BOOK, Springer-Verlag, 1999.

[3] Grimaldi, R.P., Discrete and Combinatorial Mathematics, 4th ed., Addison-Wesley, 1999.

[4] Lozansky, E., Rousseau C., Winning Solutions, Springer-Verlag, 1996.