SATISFACTION, SECRECY, AND INEQUITY IN THE PROBLEM OF FAIR DIVISION

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ABSTRACT. Two traditional criteria in the problem of fair division are proportionality, freedom from envy, and efficiency. To these we add two more: maximization of total satisfaction (the sum of all the players' evaluation of their own piece), and minimizing inequity. We show that these two criteria are inconsistent. We also show that secrecy of one's preferences, usually assumed in fair division, can be both advantageous and disadvantageous, depending upon the circumstances.

1. INTRODUCTION

A topic that has recently become popular in mathematics courses for the liberal arts major, to which Bob Bumcrot has devoted a great deal of energy, is fair division. The problem is to divide some object or set of objects among n players in such a way that all players receive their fair share in their own eyes. In this paper, we will consider the object to be divisible in a continuous way, such as a cake or a piece of land, and we will consider each player's fair share to be $1/n$. For cases in which a set of indivisible objects is to be divided up, or where each player's fair share is not necessarily $1/n$, see [1] or [2]. Different people may value different parts of the cake differently, which allow for the possibility of getting more than one's fair share.

There is a simple and elegant method for dividing a cake between two people: one person cuts and the other chooses. This method has two important qualities of fair division:

1) Proportionality: Each player is guaranteed a piece worth at least $1/n$.

2) Envy-free: Neither player prefers another player's piece.

For more than two players, the situation is more complicated. Methods were first developed after World War II by the Polish mathematicians Hugo Steinhaus, Bronislaw Knaster, and Stefan Banach. At the present time, there are envy-free, proportional procedures using a finite number of cuts for four or fewer players. For more players, the best

method so far, due to Brams and Taylor, is an envy-free, proportional procedure that can require an unbounded numbers of cuts. In practice, the remaining pieces eventually become so small that no one cares, so the procedure stops. This procedure also lacks a third important quality:
3) Efficiency: No other division is strictly better for at least one player and at least as good for the others.

2. Collective Satisfaction and Secrecy

Let the value in the eyes of player i of the piece given to player j be v_{ij}, where $0 \leq v_{ij} \leq 1$ and $\sum_{j=1}^{n} v_{ij} = 1$. It would be desirable to maximize the collective satisfaction, defined as $\sum_{i=1}^{n} v_{ii}$. If all pieces are worth $1/n$ in each player's eyes, then the total satisfaction is 1, but in general it can be greater.

For example, suppose a cake has three equal-sized parts: chocolate, vanilla, and strawberry. Suppose Alice likes chocolate and vanilla equally well, but does not care at all for strawberry. Suppose Bill likes all flavors equally. Alice makes a cut dividing the cake into one piece, containing the chocolate, and a second piece, containing the vanilla and strawberry. Alice sees both pieces as worth 1/2, but Bill sees the second piece as worth 2/3 and so chooses it. The collective satisfaction is then $1/2 + 2/3 = 7/6 \approx 1.16667$.

Fair division methods typically assume that each player does not know the other players' preferences. This is because a player can end up with a smaller piece if his or her preferences are known. In the previous example, suppose Alice knew that Bill likes all three flavors equally well. Then she can cut the cake into one piece containing all the chocolate and 49% of the vanilla, and a second part containing all the strawberry and 51% of the vanilla. Unless Bill acts of out spite for Alice taking advantage of him, he will choose the second piece, for a satisfaction of $1/3 + 0.51 * 1/3 \approx 0.5033$, while Alice ends up with a piece worth $1/2 + 0.49 * 1/2 = 0.745$. Notice that the total satisfaction is now approximately 1.2483, so Alice and Bill together are better off, but Bill is less satisfied than before.

There are other cases where lack of secrecy yields a division that is preferable to everyone. Consider the case where Alice, Bill, and Carol divide a cake with four equal-sized parts: chocolate, vanilla, strawberry, and coconut. (See figures.) All three like chocolate, only Alice likes vanilla, only Bill likes part strawberry, and only Carol likes coconut. If they all knew each other's preferences, they would realize the following obvious solution: give the vanilla to Alice, the strawberry to Bill, and the coconut to Carol, and divide the chocolate evenly between the three of them. Each player would then receive a part worth 2/3, for a collective satisfaction of 2.

In contrast, let us consider one possible outcome using the method of Brams and Taylor. (Because some of the steps are arbitrary, and the order of the players can be rearranged, different outcomes are possible.) In the first step, Alice divides the cake into three pieces that appear equal to her. There are many possibilities. Suppose the first piece consists of 2/3 of the chocolate (worth 1/3 to her) and all of the strawberry and coconut (worth nothing to her), as shown. The second piece consists of 1/3 of the chocolate and 1/3 of the vanilla. The third piece consists of 2/3 of the vanilla. Bill sees the first piece as worth 5/6, the second worth 1/6, and the third worth 0. In the second step, he trims the largest (first) piece as shown so it and the second largest piece are equal, that is, worth 1/6. In the third step, Carol puts aside the trimmings and then chooses whichever of the three pieces looks largest to her. Either the trimmed first piece or the second piece are worth 1/6; suppose she takes the second piece. Bill must take the trimmed piece if it is available, which it is in this case. Alice takes the third piece.

We repeat this process with the trimming. Alice cuts the trimming into three pieces, each containing 1/3 of the chocolate, and the third piece containing all the strawberry and coconut. Bill trims away the strawberry and coconut. Carol, then Bill, then Alice each takes one of the three identical slices.

The process can go on indefinitely as the trimmings get smaller and smaller, but in this case, Alice has no interest in the trimmings and leaves the process. Suppose that Bill and Carol divide up the trimming by Bill cutting halfway through the coconut, as shown, and Carol choosing the all-coconut piece. Then Alice's portion is worth $1/3 + (1/9)(1/2) = 7/18$. Bill's portion is worth $1/6 + (1/9)(1/2) + (1/2)(1/2) = 17/36$. Carol's portion is worth $1/6 + (1/9)(1/2) + 1/2 = 13/18$. The collective satisfaction is $19/12 \approx 1.5833$. Notice that some satisfaction is lost because Bill ends up with some vanilla, which is valuable only to Alice, and Carol ends up with some coconut, which is valuable only to Bill.

It may be that the best procedure is a division with preferences kept secret, followed by a revelation of preferences and another division, with a voting procedure for deciding between the two divisions. Such a combination procedure has not been thoroughly investigated by researchers in fair division.

3. Inequity

Another issue that has not yet been sufficiently investigated is that of inequity between the players. If $v_{ii} = v_{jj}$ for for all i and j, that is, all players' portions in their own eyes are equal, the inequity is 0. Otherwise, the inequality could be defined in various ways, such as the standard deviation of the v_{ii}.

Minimizing the inequity seems like a desirable goal. Unfortunately, it is inconsistent with the goal of maximizing the total satisfaction. Consider the example with Alice and Bill described earlier. The maximum satisfaction is achieved by giving Alice all the chocolate and vanilla, for a satisfaction of 1, and giving Bill the strawberry, for a satisfaction of 1/3, for a collective satisfaction of 4/3. This is highly inequitable, but giving Bill any more of the cake will decrease the collective satisfaction. Any time a piece is valued differently, collective satisfaction is maximized by giving that piece to the player who values it most highly.

4. Conclusions

Researchers in fair division recognize that the ideal procedure has not yet been found, namely, one that is proportional, envy-free, and efficient, and that requires only a finite number of cuts. We have further complicated matters by adding new criteria: maximizing collective satisfaction, and minimizing inequity, which conflict with each other. Most procedures rely on secrecy of preferences; we have shown that this is not necessarily a good thing. We do not have the ideal solution to recommend here, but perhaps the ideas contained here might provoke others to find a better fair division procedure.

References

[1] Steven J. Brams and Alan D. Taylor, *Fair Division: from Cake-Cutting to Dispute Resolution,* Cambridge University Press, 1996.

[2] Jack Robertson and William Webb, *Cake-Cutting Algorithms: Be Fair If You Can,* A K Peters, 1998.